Teacher notes
Topic C

The approximate Doppler formula.

This is a derivation of the approximate Doppler formula when the speed *v* of the source or observer is small compared to the speed of the wave *c*.

Sound

For sound waves and a moving source emitting sound of frequency f_0 we know that the frequency received is given by $f = f_0 \frac{c}{c - v}$ if the source is approaching. This can be rewritten as

$$f = f_0 \frac{1}{1 - \frac{v}{c}}$$

Using the binomial expansion $(1 + x)^n \approx 1 + nx$ if $x << 1$ we get

$$f \approx f_0 \left(1 + \frac{v}{c}\right)$$

or

$$f - f_0 \approx \frac{f_0 v}{c}$$

And finally

$\frac{f - f_0}{f_0} \approx \frac{v}{c}$ which is normally written as $\frac{\Delta f}{f_0} \approx \frac{v}{c}$.

Similarly, $\lambda = \frac{c}{f} = \frac{c}{f_0 \frac{1}{1 - \frac{v}{c}}} \lambda_0 = \lambda_0 \left(1 - \frac{v}{c}\right)$. Then

$\lambda - \lambda_0 = -\frac{\lambda_0 v}{c}$ or $\frac{\Delta \lambda}{\lambda_0} = -\frac{v}{c}$.

The approaching source speed *v* is positive. This means that the frequency received is greater than that emitted. The minus sign in the wavelength formula makes sure that the received wavelength is less that that emitted. But normally we just write

$$\frac{\Delta \lambda}{\lambda_0} = \frac{v}{c}$$

ignoring the minus sign and deciding from the context (observer approaching or receding) whether the wavelength decreases or increases.

For the case of the moving observer towards the source we have $f = f_0 \frac{c+v}{c}$ and so

$$f = f_0\left(1 + \frac{v}{c}\right)$$

i.e.

$\frac{\Delta f}{f_0} = \frac{v}{c}$ without any approximations. We also know that the wavelength does not change in this case: $\lambda = \frac{c+v}{f} = \frac{c+v}{f_0\left(\frac{c+v}{c}\right)} = \frac{c}{f_0} = \lambda_0$ so that $\Delta\lambda = 0$.

Light

The case of light is different. Relativistic effects lead to the Doppler formula

$$f = f_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

This applies to either a moving source or a moving observer. If the distance between the source and the observer is decreasing *v* is taken positive and negative if the distance is increasing. This is consistent with the principle of relativity: neither the source nor the observer can claim that they move and the other does not. It is the relative motion that counts.

When $v << c$ the binomial expansion gives

$$f = f_0\left(1+\frac{v}{c}\right)^{\frac{1}{2}}\left(1-\frac{v}{c}\right)^{-\frac{1}{2}} \approx f_0\left(1+\frac{v}{2c}\right)\left(1+\frac{v}{2c}\right) = f_0\left(1+\frac{v}{c}+\frac{1}{4}\frac{v^2}{c^2}\right) \approx f_0\left(1+\frac{v}{c}\right)$$

because we ignore terms of order $\frac{v^2}{c^2}$.

Again, we find $\frac{\Delta f}{f_0} \approx \frac{v}{c}$.

The wavelength also changes:

$\lambda = \frac{c}{f} = \frac{c}{f_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}} = \lambda_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$. The binomial expansion then gives

$$\lambda = \lambda_0 \left(1 - \frac{v}{c}\right)^{\frac{1}{2}} \left(1 + \frac{v}{c}\right)^{-\frac{1}{2}} \approx \lambda_0 \left(1 - \frac{v}{2c}\right)\left(1 - \frac{v}{2c}\right) = \lambda_0 \left(1 - \frac{v}{c} + \frac{1}{4}\frac{v^2}{c^2}\right) \approx \lambda_0 \left(1 - \frac{v}{c}\right)$$

The result is

$$\frac{\Delta\lambda}{\lambda_0} = -\frac{v}{c}$$

and again, we normally ignore the minus sign.

Unlike the case of sound, the wavelength changes in the case of a moving observer. This is because in sound, the moving observer measures a different speed for sound than an observer at rest in still air. In the case of light, the speed of a light wave is always *c* according to relativity. In that case, since $c = f\lambda$ or $\lambda = \frac{c}{f}$ we get by differentiation that:

$$\frac{d\lambda}{df} = -\frac{c}{f^2} = -\frac{\lambda}{f}$$

and so

$$\frac{d\lambda}{\lambda} = -\frac{df}{f} \approx -\frac{v}{c}$$

To summarize, for $v << c$,

	Sound	**Light**
	c = speed of sound in still air	*c* = speed of light in vacuum
	v = speed of source/observer	*v* = speed of source/observer
Moving source	$\frac{\Delta f}{f_0} \approx \frac{v}{c}$ and $\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c}$	$\frac{\Delta f}{f_0} \approx \frac{v}{c}$ and $\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c}$
Moving observer	$\frac{\Delta f}{f_0} \approx \frac{v}{c}$ and $\Delta\lambda = 0$	